On Representation Formulas for Long Run Averaging Optimal Control Problem

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January 21, 2015

Abstract

We investigate an optimal control problem with an averaging cost. The asymptotic behavior of the values is a classical problem in ergodic control. To study the long run averaging we consider both Cesàro and Abel means. A main result of the paper says that there is at most one possible accumulation point - in the uniform convergence topology - of the values, when the time horizon of the Cesàro means converges to infinity or the discount factor of the Abel means converges to zero. This unique accumulation point is explicitly described by representation formulas involving probability measures on the state and control spaces. As a byproduct we obtain the existence of a limit value whenever the Cesàro or Abel values are equicontinuous. Our approach allows to generalise several results in ergodic control, and in particular it allows to cope with cases where the limit value is not constant with respect to the initial condition.

Introduction

Let us consider the following control system

$$x'(t) = f(x(t), \alpha(t)),\ \alpha(t) \in A,\ t \in \mathbf{R}, \tag{1}$$

with its unique solution $x(t, x_0, \alpha)$ such that $x(0) = x_0$. Here $f : \mathbf{R}^d \times A \mapsto \mathbf{R}^d$ is supposed to be bounded and Lipschitz, and A is some compact metric space. We will denote by $\mathcal{A}$ the set of all measurable controls $\alpha : \mathbf{R} \mapsto A$.

Given a bounded cost function $l : \mathbf{R}^d \times A \mapsto \mathbf{R}$, we consider two ways of averaging the cost along the trajectory $x(t, x_0, \alpha),\ t \geq 0$,

$$(\text{Cesàro Mean}) \quad \frac{1}{T}\int_0^T l(x(s, x_0, , \alpha), \alpha(s))ds,$$

*This work has been partially supported by the French National Research Agency ANR-10-BLAN 0112.

$$\text{(Abel Mean)} \quad \lambda \int_0^{+\infty} e^{-\lambda s} l(x(s, x_0, , \alpha), \alpha(s)) ds,$$

where $\lambda > 0$ and $T > 0$. This leads to the definition of the following value functions:

$$v_T(x_0) := \frac{1}{T} \inf_{\alpha \in \mathcal{A}} \int_0^T l(x(s, x_0, t_0, \alpha), \alpha(s)) ds, \tag{2}$$

$$u_\lambda(x_0) := \lambda \inf_{\alpha \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda s} l(x(s, x_0, t_0, \alpha), \alpha(s)) ds. \tag{3}$$

The existence of the limit of v_T, as $T \to +\infty$, and of u_λ, as $\lambda \to 0^+$, is a crucial problem studied in a huge literature: We refer the reader to [1, 2, 3, 10, 17, 18, 19] and the references therein.

The works in ergodic control deal with sufficient conditions proving the existence of a limit - in the uniform convergence topology - *which is constant, i.e., which does not depend on the initial condition* x_0. These studies use different methods. For instance, the PDE method which uses the possibility of characterising the values as the unique viscosity solution of an associated Hamilton-Jacobi-Bellman equation. A typical assumption for this approach is that of coercivity of the Hamiltonian.

In the present paper we manage to consider the much harder case, where the limit is not necessarily constant (cf also [19, 18]). Our main aim is to provide a representation formula of the cluster points in the uniform convergence topology of v_T, as $T \to +\infty$, and of u_λ, as $\lambda \to 0^+$. As we will prove, there is only one possible cluster point, and the problems of the convergence of v_T and of u_λ will be reduced to the relative compactness of the families $(v_T(\cdot))_{T>0}$ and $(u_\lambda(\cdot))_{\lambda>0}$. We also stress that in [18] the authors have proved that, if v_T converges uniformly to some function then u_λ converges uniformly to the same function and conversely.

In the present work we establish two representations formulas. The first one corresponds to the case, where the cost function l does not depend on the control variable, and it is based on invariant measures for differential inclusion [4, 5]. It relates to similar questions for discrete time systems [20, 21, 22]. The second representation formula is well adapted to the more general case $(x, a) \mapsto l(x, a)$ of a cost function l depending also on the control variable, but it is based on probability measures which description with respect to the dynamical system is more involved (cf [14, 15, 16]).

Both representation formulas have a common structure: For any initial condition x_0 the only possible cluster point $u^\star(\cdot)$ is the supremum of all continuous bounded functions $w(\cdot)$ satisfying the following two conditions

i) $t \in [0, +\infty) \mapsto w(x(t, x_0, \alpha))$ is nondecreasing, for all $(x_0, \alpha) \in X \times \mathcal{A}$,

ii) $\int_X w(x) d\mu(x, a) \le \int_{X \times A} l(x, a) d\mu(x, a)$, for all $\mu \in W$,

where W is a suitable set of probability measure on $X \times A$ and $X \subset \mathbf{R}^d$ is a compact set which is supposed to be invariant with respect to the controlled equation (1).

The difference between both representation formulas concerns relation *ii*): When the cost function l does not depend on the control variable, *ii*) reduces to

ii) $\int_X w(x) d\mu(x) \le \int_X l(x) d\mu(x), \quad$ for all $\mu \in \mathcal{M}$, where $\mathcal{M}$ is the set of invariant measures of the differential inclusion. In the general case $l(x,a)$, the set W of probability measures is defined through infinitely many linear equalities.

Both formulas crucially require the existence of a compact set $X \subset \mathbf{R}^d$ invariant for (1) and that the cost l is continuous and bounded (throughout the paper we will suppose for sake of simplicity that the cost l belongs to $[0,1]$). Because the ideas of the proofs of the representations formulas for Abel and the Cesàro means are very similar, we have chosen to prove the first representation formula for the Cesàro means and the second one for the Abel means.

An important consequence of the uniqueness of the (possible) accumulation point of the values is the possibility to deduce the convergence of the values from an equicontinuity condition on the families $(v_T(\cdot))_{T>0}$ and $(u_\lambda(\cdot))_{\lambda>0}$. We discuss several nonexpansivity conditions (see [19]) which allows to obtain the equicontinuity and, thus, the convergence of the values (cf [11] for an extension to stochastic control).

Let us now describe how the paper is organised. In Section 1 we recall preliminaries and notations. Section 2 is devoted to the first representation formula with invariant measures in the case of a cost independent of the control. Section 3 concerns the general case of a cost possibly depending on the control. In Section 4 we compare briefly both obtained representation formulas. In Section 5 we discuss potential applications of the formulas and several examples. An Appendix concerning occupational measures is provided at the end of the paper.

1 Preliminaries, Assumptions and Notations

For any metric space U we will denote by $\Delta(U)$ the set of Borel probability measures on U and by $C(U)$ the set of continuous functions $\varphi : U \mapsto \mathbf{R}$. The notation $\mathcal{B}(U)$ stands for the Borel σ-algebra on U.

We will suppose throughout the article that

$$X \subset \mathbf{R}^d \text{ is compact and invariant by (1).} \tag{4}$$

Recall that this invariance means that, for all $x_0 \in X$ and all measurable control $\alpha \in \mathcal{A}$, the trajectory $t \mapsto x(t, x_0, \alpha)$ always remains in X.

We now describe the assumptions made on the coefficients f and l. Choosing as control set A a compact metric space, we suppose:

$$\begin{cases} \text{The function } l : \mathbf{R}^d \times A \longrightarrow [0,1] \text{ is continuous;} \\ \text{The function } f : \mathbf{R}^d \times A \longrightarrow \mathbf{R}^d \text{ is continuous;} \\ \exists L \geq 0, \forall (y,y') \in \mathbf{R}^{2d}, \forall a \in A,\ \|f(y,a) - f(y',a)\| \leq L\|y-y'\|; \\ \text{The set } f(x,A) \text{ is convex, for all } x \in X \end{cases} \tag{5}$$

Let $M \in \mathbf{R}$ denote a real such that $\sup_{(x,a)\in X\times A} |f(x,a)| < M$. Under assumption (5) it is well known that, for all control $\alpha \in \mathcal{A}$ and all x_0 in X, there is a unique solution $x(., x_0, \alpha) := x(\cdot)$ of (1), defined over $(-\infty, +\infty)$ and satisfying $x(0) = x_0$. We also recall that, for each positive T and λ, the value functions v_T and u_λ defined by (2) and (3), respectively, are continuous on X.

Given $T > 0$, $x_0 \in \mathbf{R}^d$ and $\alpha \in \mathcal{A}$, we associate with the solution $x(., x_0, \alpha)$ the following occupational measure $\mu_T^{x_0,\alpha} \in \Delta(X)$ defined by

$$\mu_T^{x_0,\alpha}(Q) := \frac{1}{T} meas\{s \in [0,T],\ x(s,\alpha,x_0) \in Q\}, \quad Q \in \mathcal{B}(X),$$

where $meas$ denotes the Lebesgue measure on $\mathbf{R}$. The measure $\mu_T^{x_0,\alpha}$ can be equivalently defined by the relation

$$\int_X \varphi d\mu_T^{x_0,\alpha} = \frac{1}{T}\int_0^T \varphi(x(s,x_0,\alpha))ds, \quad \varphi \in C(X).$$

We observe that $\mu_T^{x_0,\alpha} \in \Delta(X)$, where the space of probability measures $\Delta(X)$ over the compact set X is compact with respect to the topology generated by the weak convergence of measures. In the Appendix (Section 6) a description of cluster points of occupational measures is discussed.

2 First Representation Formula

In this section we first investigate the simpler case of a running cost $l(x,a) = l(x)$ which does not depend on the control variable a. In this case the cost depends only on the trajectory $x(\cdot, x_0, \alpha)$, and this makes it more convenient to use a formulation of the problem in terms of differential inclusions.

We know that under assumptions (5), any solution $x(\cdot) = x(\cdot, x_0, \alpha)$ of (1) is a solution of the differential inclusion

$$x'(t) \in F(x(t)),\ x(0) = x_0, \tag{6}$$

with

$$F(x) := \{f(x,a),\ a \in A\}.$$

Conversely, for any absolutely continuous function $x(\cdot)$ which solves (6), there exists a control $\alpha \in \mathcal{A}$ such that $x(\cdot) = x(\cdot, x_0, \alpha)$. We refer the reader to [7, 9].

For x_0 in X, we denote by $\mathcal{F}(x_0)$ the set of absolutely continuous solution of (6) with initial condition x_0, i.e., the collection of all absolutely continuous functions $x : \mathbf{R} \longrightarrow X$ such that $x(0) = x_0$ and $x'(t) \in F(x(t))$, a.e. Then we have:

$$v_T(x_0) = \frac{1}{T} \inf_{x(\cdot)\in\mathcal{F}(x_0)} \int_{t_0}^T l(x(s))ds, \quad u_\lambda(x_0) = \lambda \inf_{x(\cdot)\in\mathcal{F}(x_0)} \int_0^{+\infty} e^{-\lambda s} l(x(s))ds.$$

One advantage of the differential inclusion is its nice topological structure. Define by $\mathcal{F}$ the set of all solutions of (6) in X, i.e. $\mathcal{F} = \cup_{x_0\in X}\mathcal{F}(x_0)$. We endow $\mathcal{F}$ with the uniform topology

$$\|x\|_\infty := \sup_{t\in\mathbf{R}} |x(t)e^{-M|t|}|$$

(where $M > \sup_{(x,a)\in X\times A} |f(x,a)|$). Thanks to assumption (5) the set $\mathcal{F}$ endowed with this uniform norm is a compact metric space (cf Theorem 3.5.2 in [9]).

We also introduce the reachable map in time t associated with (1) (or (6):

$$R_t \left| \begin{array}{lll} X & \to & X \\ x_0 & \mapsto & \{x(t,x_0,\alpha),\ \alpha \in \mathcal{A}\} = \{x(t),\ x(\cdot) \in \mathcal{F}(x_0)\}. \end{array} \right.$$

2.1 Preliminaries on Invariant Measures

Let us recall some useful notions that will be used frequently in what follows (cf [4]).

We consider the following flow Φ on $\mathcal{F}$ defined for every $t \in \mathbf{R}$ by

$$\Phi_t \left| \begin{array}{ccc} \mathcal{F} & \to & \mathcal{F} \\ x(\cdot) & \mapsto & x(\cdot + t), \end{array} \right.$$

and we recall that a Borelian probability measure $p \in \Delta(\mathcal{F})$ is invariant for Φ, if and only if

$$p(\Phi_t(\Gamma)) = p(\Gamma), \qquad \text{for all } \Gamma \in \mathcal{B}(\mathcal{F}),\, t \in \mathbf{R}.$$

Definition 2.1 *Given $p \in \Delta(\mathcal{F})$, we associate it with the projected probability measure $\mu \in \Delta(X)$ defined by:*

$$\mu(B) := p(< B >),\, B \in \mathcal{B}(X), \quad \text{where } < B >:= \{x(\cdot) \in \mathcal{F},\, x(0) \in B\}.$$

The measure μ is equivalently defined by the relation $\int_X \varphi(x)d\mu(x) = \int_{\mathcal{F}} \varphi(x(0))dp(x.)$, $\varphi \in C(X)$.

If p is an invariant probability measure for Φ, then we say that the projected measure μ is a projected invariant measure on X. We denote by $\mathcal{M}$ the set of projected invariant measures on X.

Let us point out that $\mathcal{M}$ is a closed convex set of probability measures on X.

2.2 A representation Formula for Cesàro Means

Definition 2.2 *We define $\mathcal{H}$ as the set of continuous functions $w : X \to [0,1]$ satisfying the following two conditions:*

i) $t \in [0, +\infty) \mapsto w(x(t, x_0, \alpha))$ is nondecreasing, for all $(x_0, \alpha) \in X \times \mathcal{A}$;

ii) $\displaystyle\int_X w(x)d\mu(x) \leq \int_X l(x)d\mu(x)$, for all $\mu \in \mathcal{M}$.

We also introduce the function

$$v^\star(x_0) = \sup\{w(x_0),\, w \in \mathcal{H}\},\;\; x_0 \in X.$$

Let us illustrate the previous definitions with the following example.

Example 2.3 *Let X be the unit circle in $\mathbf{R}^2$, and let the uncontrolled dynamics be given by:*

$$x'(t) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; x(t). \tag{7}$$

Here, the uniform (Haar) measure on X is the unique projected invariant measure, and $\mathcal{H}$ is the set of functions $w : X \to [0,1]$ which are constant (by condition (i)) and not greater than $\frac{1}{2\pi}\int_0^{2\pi} l(e^{i\theta})d\theta$ (condition (ii)). Thus, for all x_0 in X, we have $v^*(x_0) = \frac{1}{2\pi}\int_0^{2\pi} l(e^{i\theta})d\theta$.

Remark 2.4 We say that the control problem is *leavable*, if for all x in X we have $0 \in F(x)$. This condition means that at any point $x \in X$ it is possible to stop by choosing a suitable control and stay at this point forever.

We remark that for leavable problems any probability μ in $\Delta(X)$ is a projected invariant measure, and so condition (ii) of the definition of $\mathcal{H}$ is equivalent to $w \leq l$. Then, for x_0 in X, we obtain: $v^\star(x_0) = \sup\{w(x_0),\ w : X \to [0,1]$ continuous, $w \leq l$ and such that $t \in [0,+\infty) \mapsto w(x(t,x,\alpha))$ is nondecreasing for every $\alpha \in \mathcal{A}$ and any $x \in X\}$.

Remark 2.5 We say that the problem is *controllable*, if for any states x and y in X, starting from x allows to approach to y arbitrarily closely, i.e., y belongs to the closure of the reachable set $\cup_{t\geq 0} R_t(x_0)$. In this case all functions in $\mathcal{H}$ are constant (by condition (i)), and it follows that v^* itself is constant and satisfies

$$v^*(x_0) = \inf_{\mu \in \mathcal{M}} \int_X l(x) d\mu(x), \quad x_0 \in X.$$

Example 2.6 *Let X be the unit circle in $\mathbf{R}^2$, and let the dynamics be given by*

$$x'(t) \quad = \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ a\, x(t), \quad a \in [0,1]. \tag{8}$$

Then one can generate any anticlockwise motion on X with speed varying between 0 and 1. This problem is both leavable and controllable, and so we can apply the previous remarks and obtain:

$$v^*(x_0) = \inf_{x \in X} l(x), \quad x_0 \in X.$$

Example 2.7 *We consider the two-dimensional dynamics*

$$\begin{cases} x_1'(t) = \ a(t)(1 - x_1(t)) \\ x_2'(t) = a^2(t)(1 - x_1(t)) \end{cases} \quad a(t) \in [0,1], \quad x(0) = x_0, \tag{9}$$

endowed with the cost function $l(x) = 1 - x_1(1 - x_2)$, $x = (x_1, x_2) \in \mathbf{R}^2$.

It is easy to check that $X = \{(x_1, x_2) \in [0,1]^2, x_1 \geq x_2\}$ is an invariant set, and for $x_0 \in X$ we always have $x_1'(t) \geq x_2'(t) \geq 0$. Obviously, the problem is leavable, but not controllable. However, starting from any (x_1, x_2) in X, by choosing arbitrarily small constant controls it is possible to approach arbitrarily closely to $(1, x_2)$. Thus, all functions w in $\mathcal{H}$ satisfy: $w(x_1, x_2) \leq x_2$. Moreover, as the function $w : (x_1, x_2) \to x_2$ belongs to $\mathcal{H}$, we obtain:

$$v^*(x_1, x_2) = x_2, \quad x = (x_1, x_2) \in X.$$

We observe that here the function $v^*(x_0)$ does depend on x_0 and the problem cannot be reduced to an ergodic one by changing the state space X (see cf [19] p.4).

Theorem 2.8 *Suppose that the assumptions (4) and (5) are satisfied and that, moreover, the cost function l does not depend on the control variable a. Then any accumulation point - in the uniform convergence topology on X - of $(v_T(\cdot))_{T>0}$, as $T \to \infty$, is equal to $v^\star(\cdot)$.*

Proof : Let v be an accumulation point of $(v_T(\cdot))_{T>0}$. Then, up to a subsequence, v_T converges uniformly to v as $T \to +\infty$. In order to avoid too many notations, we suppose that $v_T \to v$.

Step 1 : $v^\star(x_0) \geq v(x_0)$ For proving this, it is enough to show that $v \in \mathcal{H}$.

Let us fix arbitrarily $r < t$, $x_0 \in X$ and $\alpha \in \mathcal{A}$, and let us prove $v(x(r, x_0, \alpha)) \leq v(x(t, x_0, \alpha))$. Without loss of generality we can suppose $r = 0$. Then,

$$v_T(x(t, x_0, \alpha)) = \frac{1}{T} \inf_{\widetilde{\alpha}} \int_0^T l(x(s, x(t, x_0, \alpha), \widetilde{\alpha}))ds = \frac{1}{T} \inf_{\widetilde{\alpha}} \int_0^T l(x(s+t, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)))ds$$

where $\widetilde{\alpha}$ runs all controls in $\mathcal{A}$ and

$$\alpha \odot \widetilde{\alpha}(\cdot - t)(s) = \begin{cases} \alpha(s), & \text{if} \quad s \leq t, \\ \widetilde{\alpha}(s-t), & \text{if} \quad s > t. \end{cases}$$

Thus,

$$\begin{aligned} v_T(x(t, x_0, \alpha)) &= \frac{1}{T} \inf_{\widetilde{\alpha}} \int_t^{T+t} l(x(\sigma, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)))d\sigma \\ &\geq \frac{1}{T} \inf_{\widetilde{\alpha}} \int_0^T l(x(\sigma, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)))d\sigma - \frac{t}{T} \\ &\geq v_T(x_0) - \frac{t}{T}. \end{aligned}$$

Passing to the limit as $T \to \infty$ we get $v(x(t, x_0, \alpha)) \geq v(x_0)$. This proves that v satisfies the property i) of the definition of $\mathcal{H}$.

Now we proceed by checking that v satisfies the property ii) of the definition of $\mathcal{H}$. For this end we fix $\mu \in M$ and show that $\int_X v d\mu \leq \int_X l d\mu$.

Let $T > 0$. Since, for all $\alpha \in \mathcal{A}$,

$$v_T(x_0) \leq \frac{1}{T} \int_0^T l(x(s, x_0, \alpha))ds,$$

we have, for any $x(\cdot) \in \mathcal{F}$,

$$v_T(x(0)) \leq \frac{1}{T} \int_0^T l(x(s))ds = \frac{1}{T} \int_0^T l([\Phi_s(x(\cdot)](0))ds.$$

Now we integrate the above inequality with respect to an invariant probability measure $p \in \Delta(\mathcal{F})$ which projection is μ. By using the Fubini Theorem and the invariance property of p, we obtain

$$\begin{aligned} \int_X v_T d\mu &= \int_{\mathcal{F}} v_T(x(0))dp(x(\cdot)) \\ &\leq \frac{1}{T} \int_{\mathcal{F}} \int_0^T l([\Phi_s(x(\cdot)](0))ds dp(x(\cdot)) \\ &\leq \frac{1}{T} \int_0^T \int_{\mathcal{F}} l([\Phi_s(x(\cdot)](0))dp(x(\cdot))ds \\ &= \frac{1}{T} \int_0^T \int_{\mathcal{F}} l(x(0))dp(x(\cdot))ds \\ &= \int_{\mathcal{F}} l(x(0))dp(x(\cdot)) = \int_X l d\mu. \end{aligned}$$

Consequently, taking the limit as T tends to ∞, we get $\int_X v d\mu \leq \int_X l d\mu$. This proves that $v \in \mathcal{F}$.

Step 2: $v^\star(x_0) \leq v(x_0)$ By definition of $v^\star$ it is enough to show that $w(x_0) \leq v(x_0)$, for all $w \in \mathcal{H}$. Let us fix arbitrarily $w \in \mathcal{H}$.

For any arbitrarily given $T > 0$ and $\varepsilon > 0$, there exists a ε-optimal control $\alpha_\varepsilon \in \mathcal{A}$ such that

$$\frac{1}{T}\int_0^T l(x(s, x_0, \alpha_\varepsilon))ds \leq v_T(x_0) + \varepsilon \tag{10}$$

We consider the occupational measure on $\mu_T^{x_0,\alpha_\varepsilon} \in \Delta(X)$. This measure $\mu_T^{x_0,\alpha_\varepsilon}$ can be regarded as the projection (in the sense of Definition 2.1) of the occupational measure p_T on $\mathcal{F}$ defined by

$$\int_{\mathcal{F}} G dp_T = \frac{1}{T}\int_0^T G(\Phi_s(x(\cdot, x_0, \alpha_\varepsilon)))ds, \quad G \in C(\mathcal{F}).$$

On the other hand, relation (10) can be written as

$$\int_X l d\mu_T^{x_0,\alpha_\varepsilon} \leq v_T(x_0) + \varepsilon. \tag{11}$$

As a consequence of Prohorov's Theorem, the compactness of the spaces $\mathcal{F}$ and X implies that of $\Delta(\mathcal{F})$ and $\Delta(X)$. Consequently, there is $(p, \mu) \in \Delta(\mathcal{F}) \times \Delta(X)$ such that $(p_T, \mu_T^{x_0,\alpha_\varepsilon}) \to (p, \mu)$ weakly along a subsequence, as $T \to +\infty$. Once again, for simplicity, let us denote this weakly converging subsequence by $(p_T, \mu_T^{x_0,\alpha_\varepsilon})$. Referring to Theorem 6.2 in [4] or Lemma 6.1 of the Appendix, we know that p is an invariant probability measure and, consequently, also μ is invariant, i.e., $\mu \in \mathcal{M}$. Consequently, as $w \in \mathcal{H}$, we have $\int_X w d\mu \leq \int_X l d\mu$. On the other hand, since w satisfies the monotonicity condition i) in Definition 2.2, we have

$$\int_X w d\mu_T^{x_0,\alpha_\varepsilon} = \frac{1}{T}\int_0^T w(x(s, x_0, \alpha_\varepsilon))ds \geq w(x_0). \tag{12}$$

The relations (11) and (12), combined with $\int_X w d\mu \leq \int_X l d\mu$, allow to pass to the limit as $T \to +\infty$ and yield

$$v(x_0) + \varepsilon \geq \int_X l d\mu \geq \int_X w d\mu \geq w(x_0).$$

To finish the proof of the claimed result, it suffices to observe that $\varepsilon > 0$ has been chosen arbitrarily.

QED

Remarks 2.9 *1. The proof of Theorem 2.8 shows that the above result remains valid, when we replace $\mathcal{M}$ by the set of accumulation points (in the sense of the topology of weak convergence) of occupational measures $\mu_{T_n}^{x_0,\alpha}$ ($T_n \uparrow +\infty$):*

$$\mathcal{M}(x_0) := \{ \lim_n \mu_{T_n}^{x_0,\alpha},\, T_n \to +\infty,\, \alpha \in \mathcal{A} \}.$$

2. Observe that $\mathcal{M}(x_0) \subset \mathcal{M}$ thanks to Lemma 6.1.

3. Since $v_T(x_0) = \inf_\alpha \int_X l(x)d\mu_T^{x_0,\alpha}(x)$, we deduce from Theorem 2.8 the following "pointwise" representation formula

$$v^\star(x_0) = \inf_{\mu\in\mathcal{M}(x_0)} \int_X l d\mu. \tag{13}$$

2.3 A representation formula for Abel Means

At the end of this section we provide a representation formula for cluster points of u_λ, when $\lambda \to 0^+$.

Theorem 2.10 *We suppose that the assumptions (4) and (5) are satisfied and that, moreover, the cost function l does not depend on the control variable a. Then any accumulation point - in the uniform convergence topology - of $(u_\lambda(\cdot))_{\lambda>0}$ as $\lambda \to 0^+$, is equal to $v^\star(\cdot)$.*

We omit the proof of this result, since it is based on similar arguments than those used in Theorem 2.8, with the only real difference that now Lemma 6.2 in the appendix instead of Lemma 6.1 has to be used.

3 Second representation formula

This section is devoted to the investigation of the general case of a cost function $l(x,a)$ which may depend on the control variable a.

3.1 On the accumulation points of u_λ as $\lambda \to 0$.

We begin our studies with establishing a representation formula for the accumulation points of u_λ as $\lambda \to 0^+$. Recall that u_λ has been defined in (3).

3.1.1 Discounted Occupational Measures on $X \times A$

Given any $\lambda > 0$, x_0 and a control $\alpha \in \mathcal{A}$, we define the discounted occupational measure $\nu_\lambda^{x_0,\alpha} \in \Delta(X \times A)$ as follows:

$$\int_{X\times A} \varphi d\nu_\lambda^{x_0,\alpha}(x,a) := \lambda \int_0^{+\infty} e^{-\lambda s}\varphi(x(s,x_0,\alpha),\alpha(s))ds,\ \ \varphi \in C(X\times A),$$

and introduce the following set of discounted occupational measures over $X \times A$:

$$\Gamma_\lambda(x_0) := \{\nu_\lambda^{x_0,\alpha} \in \Delta(X\times A),\ \alpha \in \mathcal{A}\}.$$

We observe that, for all $\nu_\lambda^{x_0,\alpha} \in \Gamma_\lambda(x_0)$ and $\varphi \in C^1(X)$,

$$\int_{X\times A} \left(\nabla\varphi(x)\cdot f(x,a) + \lambda(\varphi(x_0) - \varphi(x))\right) d\nu_\lambda^{x_0,\alpha}(x,a) = 0. \tag{14}$$

Indeed

$$\int_{X\times A} \nabla\varphi(x)\cdot f(x,a)d\nu_\lambda^{x_0,\alpha}(x,a) = \lambda\int_0^{+\infty} e^{-\lambda s}\nabla\varphi(x(s,x_0,\alpha))f(x(s,x_0,\alpha),\alpha(s))ds$$
$$= \lambda\int_0^{+\infty}\left(\frac{d}{ds}[e^{-\lambda s}\varphi(x(s,x_0,\alpha))] + \lambda e^{-\lambda s}\varphi(x(s,x_0,\alpha))\right)ds$$
$$= -\lambda\varphi(x_0) + \lambda^2\int_0^{+\infty} e^{-\lambda s}\varphi(x(s,x_0,\alpha))ds = -\lambda\int_{X\times A}(\varphi(x_0)-\varphi(x))d\nu_\lambda^{x_0,\alpha}(x,a).$$

This means that

$$\Gamma_\lambda(x_0)\subset W_\lambda(x_0),$$

where

$$W_\lambda(x_0) := \{\nu\in\Delta(X\times A) : \int_{X\times A}(\nabla\varphi(x)\cdot f(x,a)+\lambda(\varphi(x_0)-\varphi(x)))\,d\nu(x,a) = 0, \text{ for all } \varphi\in C^1(X)\}.$$

Since the set $W_\lambda(x_0)$ is defined by linear equalities, it is convex. Moreover, by Prohorov's Theorem it is compact. Consequently, $co(\Gamma_\lambda(x_0))\subset W_\lambda(x_0)$, where co denotes the closed convex hull.

Now if $\nu_n\in W_{\lambda_n}(x_0)$ and $\nu_n\rightharpoonup\nu$ for some sequence $\lambda_n\to 0^+$, then, obviously, ν belongs to the set

$$(15)\quad W := \{\nu\in\Delta(X\times A) : \int_{X\times A}\nabla\varphi(x)\cdot f(x,a)d\nu(x,a) = 0, \text{ for all } \varphi\in C^1(X)\}.$$

Thus,

$$\limsup_{\lambda\to 0^+} co(\Gamma_\lambda(x_0))\subset W,$$

where $\limsup_{\lambda\to 0^+} co(\Gamma_\lambda(x_0))$ denotes the set of accumulation points[1] of all sequences $\nu_n\in co(\Gamma_{\lambda_n}(x_0))$. The above inclusion has a kind of converse as stated in the following

Lemma 3.1 *(see [15], Proposition 6.1) We have*

$$(16)\qquad \lim_{\lambda\to 0^+} d_H(co(\Gamma_\lambda(X)),W) = 0,$$

where d_H is the Haussdorff distance associated with any distance d which is consistent with the weak convergence of measures in $\Delta(X\times A)$.

Here $\Gamma_\lambda(X) = \bigcup_{x_0\in X}\Gamma_\lambda(x_0)$. Recall that the Haussdorff distance d_H between two sets M_1 and M_2 is given by

$$d_H(M_1,M_2) = \max\{\sup_{\mu\in M_1} d(\mu,M_2), \sup_{\mu\in M_2} d(\mu,M_1)\}.$$

[1]In fact, $\limsup_{\lambda\to 0^+} co(\Gamma_\lambda(x_0))$ is the upper Kuratowski limit of $co(\Gamma_\lambda(x_0))$ (cf for instance [8]).

Remark 3.2 - *The distance d on $\Delta(X \times A)$ could be any distance compatible with the weak convergence of measures, for instance a Wasserstein distance [23] or a distance defined through a dense countable family of elements of $C(X \times U)$, cf, e.g., [15].*
- The fact that relation (16) is independent of the distance d relies on the fact that it can be equivalently written as

$$\lim_{\lambda \to 0^+} co\Gamma_\lambda(X) = W,$$

where lim *denotes the Kuratowski limit of sets, see [8].*

3.1.2 Representation formula for Abel Means

Definition 3.3 *For all x_0 in X we set*

$$u^\star(x_0) := \sup\{w(x_0),\ w \in \mathcal{K}\}$$

where $\mathcal{K}$ denotes the set of all functions $w : X \to [0,1]$ which are continuous and satisfy the following conditions:
i) $t \in [0, +\infty) \mapsto w(x(t, x_0, \alpha))$ is nondecreasing, for all $(x_0, \alpha) \in X \times \mathcal{A}$;
ii) $\int_{X \times A} w(x)d\mu(x, a) \le \int_{X \times A} l(x, a)d\mu(x, a)$, for all $\mu \in W$.
Recall that the set W has been defined in (15).

Theorem 3.4 *We suppose that the assumptions (4) and (5) hold true. Then any accumulation point - in the uniform convergence topology - of $(u_\lambda(\cdot))_{\lambda>0}$ as $\lambda \to 0^+$, is equal to $u^\star(\cdot)$.*

Proof : Let us consider any accumulation point u of $(u_\lambda(\cdot))_{\lambda>0}$. Then, along a subsequence, u_λ converges uniformly to u as $\lambda \to 0^+$. In order to simplify the notation, let us suppose that $u_\lambda \to u$.

Step 1: $u^\star(x_0) \ge u(x_0)$ For proving this assertion, it suffices to show that $u \in \mathcal{K}$.

Let us fix any $r < t$, $x_0 \in X$ and $\alpha \in \mathcal{A}$, and let us show that $u(x(r, x_0, \alpha)) \le u(x(t, x_0, \alpha))$. Without loss of generality we can suppose $r = 0$. Recalling the definition of u_λ, we observe that

$$\begin{aligned} u_\lambda(x(t, x_0, \alpha)) &= \lambda \inf_{\widetilde{\alpha} \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda s} l(x(s, x(t, x_0, \alpha), \widetilde{\alpha}), \widetilde{\alpha}(s))ds \\ &= \lambda \inf_{\widetilde{\alpha} \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda s} l(x(s + t, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)), \alpha \odot \widetilde{\alpha}(\cdot - t)(s + t))ds, \end{aligned}$$

where

$$\alpha \odot \widetilde{\alpha}(\cdot - t)(s) := \begin{cases} \alpha(s), & \text{if } s \le t, \\ \widetilde{\alpha}(s - t), & \text{if } s > t. \end{cases}$$

Thus, with the help of a change of variables $\sigma = t + s$ we obtain

$$\begin{aligned}
&u_\lambda(x(t, x_0, \alpha)) = \lambda e^{\lambda t} \inf_{\widetilde{\alpha} \in \mathcal{A}} \int_t^{+\infty} e^{-\lambda\sigma} l(x(\sigma, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)), \alpha \odot \widetilde{\alpha}(\cdot - t)(\sigma)) d\sigma \\
&\geq \lambda e^{\lambda t} \inf_{\widetilde{\alpha} \in \mathcal{A}} [\int_0^{+\infty} e^{-\lambda\sigma} l(x(\sigma, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)), \alpha \odot \widetilde{\alpha}(\cdot - t)(\sigma)) d\sigma \\
&\qquad - \int_0^t e^{-\lambda\sigma} l(x(\sigma, x_0, \alpha \odot \widetilde{\alpha}(\cdot - t)), \alpha \odot \widetilde{\alpha}(\cdot - t)(\sigma)) d\sigma] \\
&\geq \lambda e^{\lambda t} \inf_{\widetilde{\alpha} \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda\sigma} l(x(\sigma, x_0, \alpha \odot \ \widetilde{\alpha}(\cdot - t)), \alpha \odot \widetilde{\alpha}(\cdot - t)(\sigma)) d\sigma - (e^{\lambda t} - 1) \\
&\geq e^{\lambda t} u_\lambda(x_0) - (e^{\lambda t} - 1).
\end{aligned}$$

Finally, passing to the limit as $\lambda \to 0^+$ we get $u(x(t, x_0, \alpha)) \geq u(x_0)$. This proves that u satisfies property i) of the definition of $\mathcal{K}$.

Now we proceed with checking that u satisfies property ii) of the definition of $\mathcal{K}$. Let us consider any $\lambda' < \lambda$ and the discounted occupational measure $\nu_{\lambda'}^{x_0,\alpha} \in \Delta(X \times A)$ associated with $(x_0, \alpha) \in X \times \mathcal{A}$ and $\lambda' > 0$. Then, taking into account that

$$u_\lambda(y) \leq \lambda \int_0^\infty e^{-\lambda r} l(x(r, y, \alpha(\cdot + s)), \alpha(r + s)) dr, \ y \in X,$$

a forward computation combined with Fubini's theorem and a change of variables yield

$$\begin{aligned}
\int_{X \times A} u_\lambda d\nu_{\lambda'}^{x_0,\alpha} &= \lambda' \int_0^\infty e^{-\lambda' s} u_\lambda(x(s, x_0, \alpha)) ds \\
&\leq \lambda' \int_0^\infty e^{-\lambda' s} \lambda \int_0^\infty e^{-\lambda r} l(x(r, x(s, x_0, \alpha), \alpha(\cdot + s)), \alpha(r + s)) dr ds \\
&= \lambda' \int_0^\infty e^{-\lambda' s} \lambda \int_0^\infty e^{-\lambda r} l(x(s + r, x_0, \alpha), \alpha(r + s)) dr ds \\
&= \lambda \int_0^{+\infty} e^{-\lambda r} \lambda' e^{\lambda' r} \int_r^{+\infty} e^{-\lambda' \sigma} l(x(\sigma, x_0, \alpha), \alpha(\sigma)) d\sigma dr \\
&= \lambda \int_0^{+\infty} e^{-\lambda r} \{\lambda' e^{\lambda' r} \int_0^{+\infty} e^{-\lambda' \sigma} l(x(\sigma, x_0, \alpha), \alpha(\sigma)) d\sigma \\
&\qquad - \lambda' e^{\lambda' r} \int_0^r e^{-\lambda' \sigma} l(x(\sigma, x_0, \alpha), \alpha(\sigma)) d\sigma\} dr \\
&= \lambda \int_0^{+\infty} e^{-\lambda r} \{\lambda'(1 + e^{\lambda' r}) \int_0^{+\infty} e^{-\lambda' \sigma} l(x(\sigma, x_0, \alpha), \alpha(\sigma)) d\sigma \\
&\quad - \lambda' \int_0^{+\infty} e^{-\lambda' \sigma} l(x(\sigma, x_0, \alpha), \alpha(\sigma)) d\sigma - \lambda' e^{\lambda' r} \int_0^r e^{-\lambda' \sigma} l(x(\sigma, x_0, \alpha), \alpha(\sigma)) d\sigma\} dr \\
&\leq \lambda \int_0^{+\infty} e^{-\lambda r} \int_{X \times A} l d\nu_{\lambda'}^{x_0,\alpha} dr + \lambda \int_0^{+\infty} e^{-(\lambda - \lambda') r} dr.
\end{aligned}$$

Consequently,

$$\int_{X \times A} u_\lambda d\nu \leq \int_{X \times A} l d\nu + \frac{\lambda'}{\lambda - \lambda'}, \qquad \text{for all } \nu \in \Gamma_{\lambda'}(X).$$

Then, by taking the limit as $\lambda' \to 0^+$ and considering Lemma 3.1, we get

$$\int_{X \times A} u_\lambda d\nu \leq \int_{X \times A} l d\nu, \qquad \text{for all } \nu \in W.$$

Letting now λ tend to 0^+, we obtain

$$\int_{X \times A} u d\nu \leq \int_{X \times A} l d\nu, \qquad \text{for all } \nu \in W,$$

and this is just condition ii) in the definition of $\mathcal{K}$.

Step 2: $u^\star(x_0) \le u(x_0)$ Let us fix any $w \in \mathcal{K}$ and prove that $w(x_0) \le u(x_0)$.

For an arbitrarily given but fixed $\varepsilon > 0$ we consider an ε-optimal control $\alpha_{\lambda,\varepsilon} \in \mathcal{A}$ for $u_\lambda(x_0)$,

$$\int_{X\times A} l d\nu_\lambda^{x_0,\alpha_{\lambda,\varepsilon}} = \lambda \int_0^\infty e^{-\lambda s} l(x(s,x_0,\alpha_{\lambda,\varepsilon}),\alpha_{\lambda,\varepsilon}(s))ds \le \varepsilon + u_\lambda(x_0). \tag{17}$$

By Prokohov's Theorem, as $\lambda \to 0^+$, $\nu_\lambda^{x_0,\alpha_{\lambda,\varepsilon}}$ converges weakly along a subsequence to some measure $\nu \in \Delta(X \times A)$. Once again we suppose for simplicity of notation that $\nu_\lambda^{x_0,\alpha_{\lambda,\varepsilon}} \rightharpoonup \nu$. By taking the limit $\lambda \to 0^+$, we deduce from (17) that

$$\int_{X\times A} l d\nu \le u(x_0) + \varepsilon. \tag{18}$$

Moreover, since $\nu_\lambda^{x_0,\alpha_{\lambda,\varepsilon}} \in \Gamma_\lambda(x_0) \subset co\Gamma_\lambda(X)$, Lemma 3.1 implies that $\nu \in W$. Consequently, from condition ii) of Definition 3.3 we have

$$\int_{X\times A} w d\nu \le \int_{X\times A} l d\nu. \tag{19}$$

On the other hand, since w satisfies condition i) of Definition 3.3, we also have

$$w(x_0) = \lambda \int_0^\infty e^{-\lambda s} w(x_0) ds \le \lambda \int_0^\infty e^{-\lambda s} w(x(s,x_0,\alpha_{\lambda,\varepsilon}),\alpha_{\lambda,\varepsilon}(s))ds = \int_{X\times A} w d\nu_\lambda^{x_0,\alpha_{\lambda,\varepsilon}}.$$

Hence, letting λ tend to 0^+, this gives

$$w(x_0) \le \int_{X\times A} w d\nu. \tag{20}$$

Finally, combining (18), (19) and (20), we obtain

$$w(x_0) \le \varepsilon + u(x_0),$$

which is just the wished conclusion, recalling that $\varepsilon > 0$ has been chosen arbitrarily.

QED

Remark 3.5 *Condition i) of Definition 3.3 of $\mathcal{K}$ can be translated equivalently in the following Hamilton-Jacobi equation satisfied by $u^\star(\cdot)$ in viscosity sense:*

$$\inf_{a\in A} \langle \nabla u^\star(x), f(x,a)\rangle = 0,\ x \in X. \tag{21}$$

Indeed, the stability result for viscosity solutions (see [10]) enables us to pass to the limit $\lambda \to 0^+$ in the Hamilton-Jacobi equation satisfied by $u_\lambda(\cdot)$, and this gives precisely (21).

3.2 On Limits of v_T as $T \to +\infty$.

This subsection is devoted to the study of a representation formula for the accumulation points of the value functions v_T as $T \to \infty$ in the case of a control l depending on the control variable (Recall the definition (2) of v_T).

3.2.1 Occupational Measures on $X \times A$

Given any $T > 0$ and $(x_0, \alpha) \in X \times \mathcal{A}$, we define the occupational measure $\mu_T^{x_0,\alpha} \in \Delta(X \times A)$ by the following relation:

$$\int_{X\times A} \varphi d\mu_T^{x_0,\alpha}(x,a) := \frac{1}{T}\int_0^T \varphi(x(s,x_0,\alpha),\alpha(s))ds, \quad \text{for all } \varphi \in C(X\times A),$$

and we introduce the set of occupational measures

$$\Gamma_T(x_0) := \{\mu_T^{x_0,\alpha} \in \Delta(X\times A),\, \alpha \in \mathcal{A}\}.$$

In particular, we observe that

$$(22) \int_{X\times A} < \nabla\varphi(x), f(x,a) > d\mu_T^{x_0,\alpha}(x,a) = \frac{1}{T}(\varphi(x(T,x_0,\alpha)) - \varphi(x_0)), \quad \varphi \in C^1(X).$$

This property allows to deduce easily that, if a sequence $\mu_n \in \Gamma_{T_n}(x_0)$ converges weakly to some μ, as $T_n \to +\infty$, then $\mu \in W$.

As in Lemma 3.1 the set W can be somehow understood as the limit of the set of occupational measures. More precisely, we have the following with $\Gamma_T(X) = \bigcup_{x_0\in X}\Gamma_T(x_0)$:

Lemma 3.6 *(See [14], Theorem 2.1)*

$$(23) \qquad \lim_{T\to+\infty} d_H(co(\Gamma_T(X)), W) = 0,$$

where d_H is the Haussdorff distance associated with any distance d which is consistent with the weak convergence of measures in $\Delta(X\times A)$.

Remark 3.7 *The relation (23) can be equivalently written as*

$$\lim_{T\to\infty} co\Gamma_T(X) = W,$$

where lim *denotes the Kuratowski limit.*

3.2.2 Representation formula for Cesàro Means

Theorem 3.8 *Let us suppose that the assumptions (4) and (5) are satisfied. Then any accumulation point in the sense of the uniform convergence topology of $(v_T(\cdot))_{T>0}$, as $T \to +\infty$, coincides with $u^\star(\cdot)$.*

We omit the proof here, because it is based on the same ideas as that for Theorem 3.4. The main difference here consists in the use of the occupational measures described in Section 3.2.1 instead of the discounted occupational measures defined in Section 3.1.1.

4 Comparison between both representation formulas

In the Sections 1 and 2 we have obtained two representations of different nature: While the first one is based on invariant measures and concerns mainly the case where l is independent of the control, the second one is based on measures which are limits of occupational measures on the product space $X\times A$. In this section we discuss the relations between the both approaches.

4.1 Invariant measure approach

A natural question is the possibility to use an approach based on invariant measures (as that in Section 2) also in the case of a running cost $l(x,a)$ which depends on the control variable. In fact, this is possible to a certain extent, if one equips the space of controls with a weak L^2-topology. Let us be more precise and discuss this question.

We define by $\mathcal{G}$ the set of all pairs $(x(\cdot),\alpha(\cdot))$ solving equation (1) on $(-\infty,+\infty)$, and we equip $\mathcal{G}$ with the product topology defined by a following norm for the $x(\cdot)$ component

$$\|x(.)\|_\infty := \sup_{t\in\mathbf{R}} |x(t)e^{-M|t|}|$$

(Recall that $M > \sup_{(x,a)\in X\times A} |f(x,a)|$) and by the L^2_{weak}-topology associated with the following L^2 norm

$$\|\alpha(\cdot)\|_{L^2} := (\int_{-\infty}^{+\infty} |\alpha(t)|^2 e^{-M|t|}dt)^{\frac{1}{2}},$$

for the $\alpha(\cdot)$ component.

From [8] we now that the space $\mathcal{G}$ is (sequentially) compact (cf Theorem 3.5.2 in [9]). Hence, since the topology of $\mathcal{G}$ is metrizable, $\mathcal{G}$ can be considered as a complete metric space and Prohorov's Theorem can be used.

Let us define the following continuous flow $\varphi = (\varphi_t)_{t\in R}$ on $\mathcal{G}$ by putting, for $t \in R$,

$$\varphi_t \begin{cases} \mathcal{G} & \to \quad \mathcal{G} \\ (x(\cdot),\alpha(\cdot)) & \mapsto \quad (x(\cdot+t),\alpha(\cdot+t)). \end{cases}$$

The main difficulty in the use of the approach of Section 2 consists in the fact that, for defining the measures, we need to consider $\int_{\mathcal{G}} l dp$ for $p \in \Delta(\mathcal{G})$, and we have to pass to the limit in such integrals. But this requires that through the function l one can define continuous functions on the space $\mathcal{G}$ endowed with the weak topology introduced above.

One possible way is to observe that the mapping

$$G_s \begin{cases} \mathcal{G} & \to \quad \mathbf{R} \\ (x(\cdot),\alpha(\cdot)) & \mapsto \quad \int_0^s \varphi(x(r),\alpha(r))dr \end{cases}$$

is continuous on $\mathcal{G}$, when $\varphi(x,a)$ is continuous in x and affine with respect to a.

Then we can associate with any $p \in \Delta(\mathcal{G})$ a family $(\nu_s)_{s>0}$ of probability measures $\nu_s \in \Delta(X\times A)$ defined by the relation

$$\int_{X\times A} \varphi(x,a)d\nu_s(x,a) := \frac{1}{s}\int_{\mathcal{G}}\int_0^s \varphi(x(r),\alpha(r))drdp(x,\alpha),$$

satisfied by all functions $\varphi : X\times A \mapsto \mathbf{R}$ continuous in x and affine in a.

Thus we are able to state and prove results similar to those of Section 2.

We define $\widehat{\mathcal{M}}$ as the set of all $\mu \in \Delta(X\times A)$ which are accumulation points - in the sense of the weak convergence of measures - for $(\nu_s)_{s>0}$ associated with all possible invariant probability measures $p \in \Delta(\mathcal{G})$ through the above relation.

Definition 4.1 *For x_0 in X we set*

$$v^\star(x_0) := \sup\{w(x_0),\, w \in \widehat{\mathcal{H}}\}$$

where $\widehat{\mathcal{H}}$ is the set of all functions $w : X \to [0,1]$ which are continuous and satisfy the following both condition:

-i) The function $t \in [0,+\infty) \mapsto w(x(t,x_0,\alpha))$ is nondecreasing, for every $(x_0,\alpha) \in X \times \mathcal{A}$;

-ii) $\displaystyle\int_{X\times A} w(x)d\mu(x,a) \le \int_{X\times A} l(x,a)d\mu(x,a), \quad \mu \in \widehat{\mathcal{M}}.$

Proposition 4.2 *Suppose that the assumptions (4) and (5) hold true and that, moreover, $a \mapsto l(x,a)$ is affine, for all $x \in X$. Then any accumulation point - in the sense of the uniform convergence topology - of $(v_T(\cdot)_{T>0}$, as $T \to \infty$, is equal to $v^\star$.*

We also have the corresponding result for the accumulation points of the sequence $(u_\lambda(\cdot))_{\lambda>0}$.

Proposition 4.3 *If the assumptions (4) and (5) are satisfied and $a \mapsto l(x,a)$ is affine, for all $x \in X$, then any accumulation point - in the sense of the uniform convergence topology - of $(u_\lambda(\cdot))_{\lambda>0}$, as $\lambda \to 0^+$, coincides with $v^\star$.*

We do not prove both above propositions because, firstly, the representation formulas (3.4) and (3.8) are more general (they do no not require that l is affine), secondly, the definition of $\widehat{\mathcal{M}}$ is less direct that those in Section 2, and last not least, the proofs are rather technical (due to the topology on $\mathcal{G}$) and this for a result weaker than (3.4) and (3.8).

4.2 The case of a cost independent of the control variable

Let us now discuss the case of a cost function l which is independent of the control variable. In this case, we have two different representation formulas in Sections 2 and 3. We will explain how the results of Section 2 can be deduced from those of Section 3 thanks to Proposition 4.4 which is stated in this subsection and concerns the relation between the set W (defined in (15)) and the set $\mathcal{M}$ given by Definition 2.1.

One can deduce from (23) and Lemma 6.1 that

$$\Pi_X(W) \subset \mathcal{M},$$

where $\Pi_X(m) \in \Delta(X)$ denotes the projection on X of a measure $m \in \Delta(X\times A)$. However, it is not clear a priori if the above inclusion is even an equality. Such an equality would imply that the both types of representation formula are in fact the same.

Let us consider $\mu \in \mathcal{M}$ and an associated invariant measure $p \in \Delta(\mathcal{F})$. Let us also fix an arbitrary $\varphi \in C^1(X)$. Then, for all $t > 0$,

$$\begin{aligned} 0 &= \int_{\mathcal{F}} \varphi(\Phi_t(x.)(0))dp(x.) - \int_{\mathcal{F}} \varphi(x(0)\cdot)dp(x.) \\ &= \int_{\mathcal{F}} \varphi(x(t))dp(x.) - \int_{\mathcal{F}} \varphi(x(0))dp(x.). \end{aligned}$$

Then, dividing by t and letting $t \to 0^+$, we obtain by the dominated convergence theorem that

$$0 \leq \int_{\mathcal{F}} \sup_{a\in A} \langle \nabla\varphi(x(0)), f(x(0),a)\rangle dp(x.) = \int_X \sup_{a\in A} \langle \nabla\varphi(x), f(x,a)\rangle d\mu(x).$$

By replacing in the above relation φ by $-\varphi$, we obtain[2]

$$\int_X \inf_{a\in A} \langle \nabla\varphi(x), f(x,a)\rangle d\mu(x) \leq 0 \leq \int_X \sup_{a\in A} \langle \nabla\varphi(x), f(x,a)\rangle d\mu(x), \quad \mu \in \mathcal{M}.$$

Thus, for the particular case where $f(x,a)$ does not depend on a, we have

$$\int_X \langle \nabla\varphi(x), f(x)\rangle d\mu(x) = 0, \quad \mu \in \mathcal{M},$$

and, consequently, $\Pi_X(W) = \mathcal{M}$. In fact, this relation is valid in a more general context, as it is stated in the following

Proposition 4.4 *Let the set W be defined by (15) and the set $\mathcal{M}$ be given by Definition 2.1. Then*

$$\Pi_X(W) = \mathcal{M}. \tag{24}$$

Before proving the above Proposition we state two auxiliary results which proofs are postponed to the end of the section.

Lemma 4.5 *For every $x(\cdot) \in \mathcal{F}$ there exists a measurable control $\alpha_{x(\cdot)} \in \mathcal{A}$ such that, for almost all $t \in \mathbf{R}$, we have $x'(t) = f(x(t), \alpha_{x(\cdot)}(t))$ and such that, moreover, the mapping*

$$\begin{cases} \mathcal{F} & \to & \mathcal{A} \\ x(\cdot) & \mapsto & \alpha_{x(\cdot)} \end{cases}$$

is measurable, where $\mathcal{A}$ is endowed with the $L^2(e^{-m_A|t|}dt)$ topology[3] and $\mathcal{F}$ is equipped with the uniform topology described in Section 2.

Lemma 4.6 *For all t, t' in $\mathbf{R}$ and all $\varphi \in C^1(X)$ we have*

$$\int_{\mathcal{F}} \int_t^{t'} \langle \nabla\varphi(x(s)), f(x(s), \alpha_{x(\cdot)}(s))\rangle ds dp(x.) = 0,$$

for any invariant measure $p \in \Delta(\mathcal{F})$.

Proof of Proposition 4.4: Since we already know that $\Pi_X(W) \subset M$, we only need to prove the converse inclusion. Let us consider an arbitrary $\mu \in \mathcal{M}$ and let $p \in \Delta(\mathcal{F})$ be an associated invariant measure. We want to show that $\mu \in \Pi_X(W)$.

---

[2] This is also related to the relation $0 \in \int_X F(x)d\mu(x)$ proved in Proposition 4.2 of [4] for an invariant measure $\mu \in \mathcal{M}$.

[3] Here m_A denotes an upper bound of the distance function $d(a,0)$ on A, where 0 is a reference point of the compact set A and a runs A.

For this end, for any $h>0$, let $\gamma_h \in \Delta(X\times A)$ be the probability measure defined by the relation

$$\int_{X\times A}\psi d\gamma_h=\int_{\mathcal{F}}\frac{1}{h}\int_0^h \psi(x(s),\alpha_{x(\cdot)}(s))dsdp(x.),\quad \psi\in C(X\times A).$$

We observe that the above integral is well defined thanks to Lemma 4.5. Moreover, by Prokhorov's Theorem we know that there exists an accumulation point $\gamma\in\Delta(X\times A)$ of γ_h as $h\to 0^+$. By abuse of notation we suppose that $\gamma_h \rightharpoonup \gamma$ weakly.

We claim that $\Pi_X(\gamma)=\mu$. Indeed, for any fixed $\varphi\in C(X)$ we have thanks to the dominated convergence theorem

$$\begin{aligned}\int_X \varphi d\Pi_X(\gamma) &= \int_{X\times A}\varphi(x)d\gamma(x,a)=\lim_{h\to 0^+}\int_{X\times A}\varphi(x)d\gamma_h(x,a)\\ &= \lim_{h\to 0^+}\int_{\mathcal{F}}\frac{1}{h}\int_0^h\varphi(x(s))dsdp(x.)=\int_{\mathcal{F}}\varphi(x(0))dp(x(\cdot))\\ &= \int_X\varphi(x)d\mu(x).\end{aligned}$$

Now, again for $\varphi\in C^1(X)$, we observe that

$$\begin{aligned}\int_X\langle\nabla\varphi(x),f(x,a)\rangle d\gamma(x,a) &= \lim_{h\to 0^+}\int_{X\times A}\langle\nabla\varphi(x),f(x,a)\rangle d\gamma_h(x,a)\\ &= \lim_{h\to 0^+}\int_{\mathcal{F}}\frac{1}{h}\int_0^h\langle\nabla\varphi(x(s)),f(x(s),\alpha_{x(\cdot)}(s))\rangle dsdp(x.)=0\end{aligned}$$

due to Lemma 4.6. But, $\int_X\langle\nabla\varphi(x),f(x,a)\rangle d\gamma(x,a)=0$, for all $\varphi\in C^1(X)$, means that $\gamma\in W$.

Hence $\mu=\Pi_X(\gamma)\in\Pi_X(W)$, and the proof is complete.

QED

Let us now come to the proof of the auxiliary lemmas.

Proof of Lemma 4.5 : It is easy to show that the function

$$D\left\{\begin{array}{lll}\mathcal{F} & \to & L^2(\mathbf{R}\to X;e^{-m_A|t|}dt)\\ x(\cdot) & \mapsto & x'(\cdot)\end{array}\right.$$

has a closed graph with respect to the $\mathcal{F}\times L^2(\mathbf{R}\to X;e^{-M|t|}dt)$ product topology. Hence, it is in particular measurable. Let us now show that the set-valued map

$$E\left\{\begin{array}{lll}Graph(E) & \to & L^2(\mathbf{R}\to A;e^{-m_A|t|}dt)\\ (x(\cdot),x'(\cdot)) & \mapsto & \{\alpha(\cdot)\in L^2(e^{-m_A|t|}dt),\ x'(t)=f(x(t),\alpha(t)),\ \text{for a.e. } t\in\mathbf{R}\}\end{array}\right.$$

is measurable with respect to the $(\mathcal{F}\times L^2(\mathbf{R}\to X;e^{-M|t|}dt))-L^2(\mathbf{R}\to A,e^{-m_A|t|}dt)$ topology. We know that E has nonempty images. Once again the measurability of E is a consequence of the fact that E has a closed graph, and this we will prove now.

For this end we consider a sequence $(x_n(\cdot),x_n'(\cdot),\alpha_n(\cdot))\in Graph(E)$ and we suppose that $(x_n(\cdot),x_n'(\cdot),\alpha_n(\cdot))$ converges to some $(x(\cdot),y(\cdot),\alpha(\cdot))$ in the $\mathcal{F}\times L^2(\mathbf{R}\to X;e^{-M|t|}dt)\times L^2(\mathbf{R}\to A;e^{-m_A|t|}dt)$ product topology. We want to prove that $\alpha(\cdot)\in$

$E(x(\cdot), y(\cdot))$. As $Graph(D)$ is closed, we know already that $y(\cdot) = x'(\cdot)$. On the other hand, since $\alpha_n(\cdot)$ converges to $\alpha(\cdot)$ in $L^2(\mathbf{R} \to A; e^{-m_A|t|}dt)$, it converges also in measure dt. Let us denote by $\omega_f(\cdot)$ the modulus of continuity of $f(x, a)$ with respect to the variable a on the compact $X \times A$. Then, for all $t \in \mathbf{R}$,

$$
\begin{aligned}
&|x(t) - x(0) - \int_0^t f(x(s), \alpha(s))ds| \\
&\leq |x_n(t) - x_n(0) - \int_0^t f(x_n(s), \alpha_n(s))ds| + |x_n(t) - x(t)| + |x(0) - x_n(0)| \\
&\quad + \int_0^t |f(x_n(s), \alpha_n(s)) - f(x(s), \alpha_n(s))ds| + \int_0^t |f(x(s), \alpha_n(s)) - f(x(s), \alpha(s))|ds \\
&\leq 2\|x_n(\cdot) - x(\cdot)\|_{\mathcal{F}} e^{M|t|} + K\|x_n(\cdot) - x(\cdot)\|_{\mathcal{F}} \int_0^t e^{M|s|}ds + \int_0^t \omega_f(|\alpha_n(s)) - \alpha(s)|)ds,
\end{aligned}
$$

where we have used for the latter estimate the Lipschitz continuity of f in x and its uniform continuity in a. We notice that, thanks to the dominated convergence theorem, the latter integral converges to zero, as $n \to +\infty$, and we also observe that all the other terms vanish in the limit. Consequently,

$$
x(t) = x(0) + \int_0^t f(x(s), \alpha(s))ds, \text{ for all } t,
$$

but this means that $\alpha(\cdot) \in E(x(\cdot), x'(\cdot))$. Therefore, the graph of E is closed.

This implies that the set valued map

$$
Z \left\{ \begin{array}{lll} \mathcal{F} & \to & L^2(\mathbf{R} \to A; e^{-m_A|t|}dt) \\ x(\cdot) & \mapsto & \{\alpha(\cdot) \in L^2(e^{-m_A|t|}dt),\ x'(t) = f(x(t), \alpha(t)) \text{ for a.e. } t \in \mathbf{R}\} \end{array} \right.
$$

is measurable, since it is just a composition between the two measurable maps E and D. Finally, the measurable selection theorem (cf, e.g., Theorem 8.1.3 in [8]) enables us to choose a measurable selection $x(\cdot) \mapsto \alpha_{x(\cdot)}(\cdot)$ of Z. The proof is complete.

QED

Proof of Lemma 4.6: Let $p \in \Delta(\mathcal{F})$ be an arbitrary invariant measure. Let us fix any $t, t' \in \mathbf{R}$ with $t < t'$ and let $\varphi \in C^1(X)$. Since p is an invariant measure, we know that

$$
\int_{\mathcal{F}} \frac{\varphi(x(s+h)) - \varphi(x(s))}{h} dp(x(\cdot)) = 0, \tag{25}
$$

for all $h > 0$ and all $s \in [t, t']$. On the other hand, as for all $x(\cdot) \in \mathcal{F}$, ds-a.e. in $[t, t']$,

$$
\lim_{h \to 0^+} \frac{\varphi(x(s+h)) - \varphi(x(s))}{h} = \langle \nabla\varphi(x(s)), f(x(s), \alpha_{x(\cdot)}(s)) \rangle,
$$

we get from the dominated convergence theorem that, for all $x(\cdot) \in \mathcal{F}$,

$$
\lim_{h \to 0^+} \int_t^{t'} \frac{\varphi(x(s+h)) - \varphi(x(s))}{h} ds = \int_t^{t'} \langle \nabla\varphi(x(s)), f(x(s), \alpha_{x(\cdot)}(s)) \rangle ds.
$$

Thus,

$$\int_{\mathcal{F}}\int_t^{t'} \langle \nabla\varphi(x(s)), f(x(s),\alpha_{x(\cdot)}(s))\rangle ds dp(x(\cdot))$$
$$= \int_{\mathcal{F}} \lim_{h\to 0^+}\int_t^{t'} \frac{\varphi(x(s+h))-\varphi(x(s))}{h} ds dp(x(\cdot)).$$

Consequently, applying Fatou's Lemma and Fubini's Theorem, we obtain in view of (25)

$$\begin{aligned}&\int_{\mathcal{F}}\int_t^{t'} \langle \nabla\varphi(x(s)), f(x(s),\alpha_{x(\cdot)}(s))\rangle ds dp(x(\cdot))\\ &\quad\le \liminf_{h\to 0^+}\int_{\mathcal{F}}\int_t^{t'} \frac{\varphi(x(s+h))-\varphi(x(s))}{h} ds dp(x(\cdot))\\ &\quad= \liminf_{h\to 0^+}\int_t^{t'}\int_{\mathcal{F}} \frac{\varphi(x(s+h))-\varphi(x(s))}{h} dp(x(\cdot)) ds = 0.\end{aligned}$$

Hence, for all $\varphi \in C^1(X)$,

$$\int_{\mathcal{F}}\int_t^{t'} \langle \nabla\varphi(x(s)), f(x(s),\alpha_{x(\cdot)}(s))\rangle ds dp(x(\cdot)) \le 0.$$

By changing φ into $-\varphi$, we complete the proof.

QED

5 Applications and Examples

One of the main advantages of the representation formulas provided in the previous sections lies in the fact that the convergence of averaging values is deduced from equicontinuity.

Corollary 5.1 *We suppose that the assumptions (4) and (5) are satisfied. Then, if the family $(v_T(\cdot))_{T>0}$ is equicontinuous, it converges uniformly to $u^\star(\cdot)$, as $T\to\infty$. The same holds true for the family $(u_\lambda(\cdot))_{\lambda>0}$: Its equicontinuity implies its uniform convergence to $u^\star(\cdot)$, as $\lambda\to 0^+$.*

Indeed, from the Arzelà-Ascoli Theorem we deduce that $(v_T(\cdot))_{T>0}$ and $(u_\lambda(\cdot))_{\lambda>0}$ are relatively compact, and by the Theorems 2.8 and 2.10 we obtain the uniform convergence of these both families of functions to $u^\star(\cdot)$.

Let us illustrate this observation in several cases.

5.1 Nonexpansivity

Corollary 5.2 *(Nonexpansivity [19]) Let us suppose that the assumptions (4), (5) as well as the following nonexpansivity condition hold true: There is some $c \in R+$ such that, for all $x_1, x_2 \in X$,*

$$\sup_{a_1\in A}\inf_{a_2\in A}\max\{\langle x_1-x_2, f(x_1,a_1)-f(x_2,a_2)\rangle, |l(x_1,a_1)-l(x_2,a_2)|-c|x_1-x_2|\}\le 0.$$

Then both $v_T(\cdot)$ and $u_\lambda(.)$ converge uniformly to $u^\star$, as $T\to\infty$ and $\lambda\to 0^+$, respectively.

Under the above nonexpansivity condition [19] proves the existence of a limit. A crucial argument in the proof enables to deduce the equicontinuity of the families $(v_T(\cdot))_{T>0}$ and $(u_\lambda(\cdot))_{\lambda>0}$ from the above nonexpansivity condition (26). Finally, from Corollary 5.1 we obtain the uniform convergence to $v^\star(\cdot)$.

Let us also observe that for a Lipschitz cost $l(x)$ independent of the control variable a the nonexpansivity condition reduces to

$$\sup_{a_1\in A}\inf_{a_2\in A}\langle x_1-x_2, f(x_1,a_1)-f(x_2,a_2)\rangle \leq 0,\ x_1,x_2\in X. \tag{26}$$

Indeed the nonexpansivity condition of Corollary 5.2 is obtained from (26) by choosing as constant c a Lipschitz constant of l.

Let us also point out that [19] studies several extensions of the nonexpansivity condition (26). However, for all these extensions we obtain the equicontinuity of the families $(v_T(\cdot))_{T>0}$ and $(u_\lambda(\cdot))_{\lambda>0}$ and, hence, with the help of the Theorems 2.8 and 2.10, their uniform convergence to $u^\star(\cdot)$. We also mention the paper [12], where equicontinuity and monotone convergence are proved by PDE arguments.

The interest of the nonexpansivity property lies in the fact that under this condition the limit $u^\star(\cdot)$ may depend on the initial condition x_0. We refer the reader to several examples discussed in Section 2.2 of [19]. Unlike the nonexpansivity property, the following stronger condition - called dissipativity condition - implies that the limit $u^\star(\cdot)$ is a constant independent of x_0 (see [6]):

$$\sup_{a_1\in A}\inf_{a_2\in A} < x_1-x_2, f(x_1,a_1)-f(x_2,a_2) >\leq -C\|x_1-x_2\|^2,$$

for all $x_1,x_2\in X$ and some constant $C>0$.

This convergence to a constant phenomenon will be met also in the case discussed in the following subsection.

5.2 Coercivity of the Hamiltonian and controllability

Let us focus our consideration upon the Abel means u_λ, when the cost function l is Lipschitz (Remark that in this case also u_λ is Lipschitz). Our aim is to discuss and to comment a well known PDE approach with the help of the results obtained in the present paper (See [1, 2, 3], [10], [17] and the references therein).

A typical assumption in this pde approach is the coercivity of the Hamiltonian :

$$\lim_{|p|\to\infty} H(x,p)=+\infty. \tag{27}$$

The PDE approach is based on the fact that u_λ is the unique viscosity solution of the Hamilton-Jacobi equation

$$V(x)+H(x,\frac{1}{\lambda}\nabla V(x))=0, \tag{28}$$

with the Hamiltonian

$$H(x,p):=\max_{a\in A}\{\langle -f(x,a),p\rangle - l(x,a)\},\quad x,p\in\mathbf{R}^d.$$

Let us explain roughly[4] this method in view of our result Theorem 2.10. Since, thanks to our assumptions u_λ is bounded by some constant C independent of λ, the coercivity assumption (27) together with (28) enables us to obtain that $\frac{1}{\lambda}\nabla u_\lambda$ is bounded by some constant M independent of λ. This implies not only that u_λ is equi-Lipschitz (and thus, by Corollary 5.1, $\lim_{\lambda\to 0+} u_\lambda = u^\star$), but also that the Lipschitz constant of u_λ is bounded by $M\lambda$ and, hence, its convergence to zero. It follows that the function $u^\star$ must be constant.

It is worth pointing out that in this case equation (21) - which is deduced from condition i) of the definition of $\mathcal{K}$ - does not bring any information on the constant $u^\star$. But condition ii) of the definition of $\mathcal{K}$ does imply that the constant $u^\star$ is given by

$$u^\star = \inf_{\mu\in W} \int_{X\times A} l d\mu,$$

which can be compared to results obtained through the weak KAM theory [13].

Finally, let us mention that the coercivity (27) of the Hamiltonian is usually deduced from controllability conditions on the control system (1) (cf [10]). Such a typical controllability assumption is that of the existence of an $\omega > 0$ such that, for all $x \in \mathbf{R}^d$,

$$B(0,\omega) := \{y \in \mathbf{R}^d,\ |y| < \omega\} \subset F(x) = \{f(x,a),\ a \in A\}. \tag{29}$$

Similar ideas than those described above for Abel means can be used to show that under the coercivity condition (27) the values v_T converge uniformly to a constant, as $T \to +\infty$.

5.3 Examples

We have seen in examples that the limit $u^\star$ may not be constant. Let us revisit the examples 2.3 and 2.7 from Section 2.

Example 2.3 Here the coercivity condition (27) is not fulfilled. Indeed, it cannot be satisfied, when the dynamics and the cost function do not depend on the control. But, on the other hand, if the cost function $l : x \in \mathbf{R}^2 \mapsto l(x) \in [0,1]$ is Lipschitz and independent of a, Corollary 5.2 shows that, choosing as c the Lipschitz constant of l, we have the convergence of u_λ and v_T, as $\lambda \to 0+$ and $T \to +\infty$, respectively.

Example 2.7 By explicit computations it can be shown that v_T converges to $v^\star(x) = x_2$ (cf [19] p.4). However, also a variant of Corollary 5.2 can be used here. Indeed, it is easy to prove that here, for the given example,

$$\sup_{a_1\in A}\inf_{a_2\in A} \langle f(x_1,a_1), \frac{\partial}{\partial x_1}\Delta(x_1,x_2)\rangle + \langle f(x_2,a_2), \frac{\partial}{\partial x_2}\Delta(x_1,x_2)\rangle \le 0 \tag{30}$$

with

$$\Delta(x,y) := |x_1 - y_1| + |x_2 - y_2|.$$

[4]This explanation would be correct, if u_λ were of class C^1. However, since it is only continuous, a rigorous proof would require detailed a priori estimations of the super and subdifferentials of u_λ; cf, e.g., [17].

But this implies that the families $(v_T(\cdot))_{T>0}$ and $(u_\lambda(\cdot))_{\lambda>0}$ are equicontinuous (cf Proposition 3.6 in [19]), and, thus, they both converge to $u^\star(\cdot)$ uniformly on X.

This example illustrates that the equicontinuity can be obtained through various nonexpansivity conditions. The nonexpansivity condition (26) corresponds to the case, where Δ is the Euclidean norm.

6 Appendix

Let us consider a flow Φ defined on a complete metric space P. Recall that a flow Φ is a function $\mathbf{R} \times P \ni (t,p) \mapsto \Phi_t(p) \in P$ satisfying the three following conditions:

a) Φ is continuous,

b) $\Phi(0,p) = p$, for all $p \in P$,

c) $\Phi(t+s,p) = \Phi(t,\Phi(s,p))$, for all $p \in P$ and all reals t, s.

We recall the following well-known Lemma which proof is given for the reader convenience (See, e.g., Lemma 5.4 in [4] or Proposition 4.3 in [5]).

Lemma 6.1 *Suppose that the complete metric space P is compact. We fix $q \in P$ and we define for all $T > 0$ the occupational measure μ_T^q as follows*[5]

$$\int_P \varphi d\mu_T^q := \frac{1}{T}\int_0^T \varphi(\Phi_s(q))ds, \qquad \textit{for all } \varphi \in C(P).$$

If for some sequence $T_n \to \infty$ we have the weak convergence of $\mu_{T_n}^q$ to some measure μ, then μ is an invariant measure for the flow Φ.

Proof : In order to prove that μ is an invariant measure, we fix any $\varphi \in C(P)$ and $t > 0$, and we show that

$$\int_P \varphi(p)d\mu(p) = \int_P \varphi(\Phi_t(p))d\mu(p). \tag{31}$$

We remark that, thanks to the flow condition c) and a change of variable,

$$\begin{aligned}
&\int_P \varphi(\Phi_t(p))dm_{T_n}^q(p)\\
&\quad = \frac{1}{T_n}\int_0^{T_n} \varphi(\Phi_t(\Phi_s(q)))ds\\
&\quad = \frac{1}{T_n}\int_0^{T_n} \varphi(\Phi_{t+s}(q))ds = \frac{1}{T_n}\int_t^{t+T_n} \varphi(\Phi_\sigma(q))d\sigma\\
&\quad = \frac{1}{T_n}\int_0^{T_n} \varphi(\Phi_\sigma(q))d\sigma + \frac{1}{T_n}\int_{T_n}^{t+T_n} \varphi(\Phi_\sigma(q))d\sigma - \frac{1}{T_n}\int_0^{t} \varphi(\Phi_\sigma(q))d\sigma\\
&\quad = \int_P \varphi(p)d\mu_{T_n}^q(p) + \frac{1}{T_n}\int_{T_n}^{t+T_n} \varphi(\Phi_\sigma(q))d\sigma - \frac{1}{T_n}\int_0^{t} \varphi(\Phi_\sigma(q))d\sigma
\end{aligned}$$

Thus,

$$|\int_P \varphi(\Phi_t(p))d\mu_{T_n}^q(p) - \int_P \varphi(p)d\mu_{T_n}^q(p)| \le \frac{2t}{T_n}\|\varphi\|_\infty.$$

[5] The occupational measure μ_T^q can be equivalently defined by $\mu_T^q(Q) := \frac{1}{T} meas\{\tau \in [0,T],\ \Phi_\tau(q) \in Q\}$, $Q \in \mathcal{B}(P)$.

Finally, passing in the above relation to the limit as $n \to +\infty$, we obtain with

$$|\int_P \varphi(\Phi_t(p))d\mu(p) - \int_P \varphi(p)d\mu(p)| = 0$$

the wished equation (31).

QED

For discounted occupational measures, we have the following result which proof is very similar to that of the preceding Lemma.

Lemma 6.2 *Let P be a compact complete metric space. For all $q \in P$ and $\lambda > 0$ we define the occupational measure ν_λ^q by the following relation:*

$$\int_P \varphi d\nu_\lambda^q := \lambda \int_0^{+\infty} e^{-\lambda s}\varphi(\Phi_s(q))ds, \quad \text{for all } \varphi \in C(P).$$

Then, if for some sequence $\lambda_n \to 0^+$ we have the weak convergence of $\nu_{\lambda_n}^q$ to some measure ν, then this measure ν is invariant for the flow Φ.

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